Boundary effects in a random neighbor model of earthquakes

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We introduce spatial inhomogeneities (boundaries) in a random neighbor version of the Olami, Feder, and Christensen model [Phys. Rev. Lett. **68**, 1244 (1992)] and study the distributions of avalanches starting both from the bulk and from the boundaries of the system. Because of their clear geophysical interpretation, two different boundary conditions have been considered (named *free* and *open*, respectively). In both cases the bulk distribution is described by the exponent $\tau \simeq \frac{3}{2}$. Boundary distributions are instead characterized by two different exponents $\tau' \simeq \frac{3}{2}$ and $\tau' \simeq \frac{7}{4}$, for free and open boundary conditions, respectively. These exponents indicate that the mean-field behavior of this model is correctly described by a recently proposed inhomogeneous form of the critical branching process. [S1063-651X(98)06803-2]

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Some ten years ago Bak, Tang, and Wiesenfeld (BTW) introduced the concept of self-organized criticality (SOC) [1]. The term refers to the tendency of a large class of extended dynamical systems to spontaneously organize into a dynamical critical state characterized by long range correlations, in both space and time. Many models displaying SOC have been introduced and used to describe, for instance, sandpiles [1], earthquakes [2], and biological evolution [3]. The common feature of all these models is the simplicity of the local dynamical rules, to be contrasted to the global complex structures (e.g., fractal) that emerge as a result of the continued local interactions. A model that has recently attracted much attention in the literature is that proposed by Olami, Feder, and Christensen (OFC) [2], to mimic earthquake dynamics. This model has revealed itself to be particularly challenging to understand and has raised many fundamental questions on the very nature of SOC [2,4–6].

The OFC model is a coupled-map lattice model, where with each site (i,j) of a square lattice, there is associated a continuous ''energy'' E_{ij}, initially fixed at a random value in the interval $(0,E_c)$. All the energies are increased uniformly and simultaneously at the same speed, until one of them reaches the threshold value E_c and becomes unstable ($E_{ij} \geq E_c$). The uniform driving is then stopped and an ''earthquake'' starts:

$$E_{i,j} \geq E_c \Rightarrow \begin{cases} E_{i,j} \rightarrow 0 \\ E_{nn} \rightarrow E_{nn} + \alpha E_{i,j}, \end{cases} \quad (1)$$

where nn denotes the set of nearest-neighbor sites of (i,j) and $\alpha \in [0,\frac{1}{4}]$ is a parameter controlling the level of conservation of the dynamics ($\alpha = \frac{1}{4}$ corresponds to the conservative case). The ''toppling'' rule (1) can possibly produce a chain reaction, which ends when all sites are stable again

($E_{kl} < E_c$, $\forall kl$). The uniform growth then starts again. Boundary conditions imply that sites close to boundaries of the system have a smaller coordination number (except in the periodic case) and, in principle, also a different value for α (α_{BC}). Open boundary conditions are generally set. In this case $\alpha_{BC} = \alpha$. The absence of a characteristic length scale in the system is reflected in the behavior of the probability $P(s)$ that an earthquake involves s sites. Indeed, in the stationary state, $P(s) \propto s^{-\tau}$, where τ is a critical exponent.

Contrary to the BTW sandpile model [7], the OFC model is believed to be self-organized critical even when the dynamics is nonconserving ($\alpha < \frac{1}{4}$) [2,4–6]. In a sense this is a rather surprising, not yet fully understood result, which possibly implies a peculiar mechanism leading to SOC, different from that of conserved models. Moreover, the avalanche distribution exponent τ is nonuniversal: it seems to depend on the conservation parameter α and even on the type of boundary conditions [2]. One of the peculiarities of this model is the role played by boundaries. Besides being ''sinks'' for the energy in excess in the system (which, moreover, can be dissipated even in the bulk, due to the nonconserving dynamics), they act as inhomogeneities that frustrate the natural tendency of the model to synchronize. This is believed to be the fundamental mechanism producing SOC in this system [5,6]. Indeed, it has been shown that, with periodic boundary conditions and for sufficiently small values of the conservation parameter α ($\alpha \simeq 0.18$), the system reaches a strictly periodic state, in which each avalanche involves just one site [4–6]. For larger values of α the situation is slightly more complicated with multiple topplings involved in a single avalanche, but the avalanches are still localized and criticality is not observed. Open boundaries create a discontinuity in a homogeneous system and break the periodic state it would otherwise reach. In Ref. [6] it was suggested that sites close to the boundaries start to organize themselves first, building up long range correlations. The critical region grows with time until, in the stationary state, it invades the whole lattice.

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Despite the simplicity of local dynamical rules in SOC systems, analytical approaches have rarely been successful (for an exception, see, e.g., Ref. [8]). This applies, in particular, to models of the OFC type. Most of the results achieved so far have been obtained through computer simulations. To overcome this limitation it has sometimes been useful to consider random neighbor (RN) models [9–11], where each site interacts with randomly chosen sites instead of its nearest neighbors on the lattice. This considerably simplifies the problem, though retaining some essential features of the original model. Moreover, RN models can be seen as mean-field descriptions of their nearest-neighbor counterparts, since spatial correlations are neglected.

The RN versions of the BTW sandpile model [1] and the Bak and Sneppen (BS) evolutionary model [3] have been solved exactly in Ref. [9] and Ref. [10], respectively. The solutions show that these models become equivalent to a critical branching process. Accordingly, the avalanche size distribution exponent is $\tau = \frac{3}{2}$.

The RN version of the OFC model [11] is described by the following toppling rule:

$$E_{i,j} \geqslant E_c \Rightarrow \begin{cases} E_{i,j} \rightarrow 0 \\ E_{rn} \rightarrow E_{rn} + \alpha E_{i,j}, \end{cases}$$

where rn stands for four sites chosen randomly in a finite lattice box. Open boundary conditions are implemented by requiring that sites at the box boundary can collect an infinite amount of energy without toppling. In this model all bulk sites are equivalent and, therefore, no geometrical meaning can be attached to boundaries.

A numerical investigation of this version of the RN OFC model gave evidence that in the stationary state avalanches are power law distributed in a whole range of α values ($\alpha_c \lesssim \alpha \leqslant 1/4$; $\alpha_c \simeq 0.225$), with an exponent $\tau \simeq 3/2$. These results have been questioned by some recent works aimed at an exact analytical control of the RN model [12,13]. By studying a continuum-time equation it was deduced that avalanches are localized as soon as $\alpha < 1/4$, although the mean avalanche size grows exponentially fast as dissipation tends to zero. Anyway, as far as the description of boundary effects is concerned, this approach is even less satisfactory. In fact no ''boundary'' dissipation is explicitly introduced in Refs. [12,13], which, strictly, would make the system ''explode'' in the conservative limit ($\alpha \rightarrow 1/4$).

The main purpose of the present work is that of providing a formulation in which the OFC model is still of a RN nature, while allowing a meaningful distinction between the boundary and bulk of the system. This formulation, which can be handled numerically, is worth analyzing in view of the important role boundary inhomogeneities are expected to play in the nearest-neighbor OFC model. Moreover, our model, introducing the notion of position in the system, constitutes a substantial improvement of the standard RN one, just like a Landau-Ginzburg approach compared to the Weiss theory of ferromagnetism.

In order to introduce a proper inhomogeneity due to boundary effects, we have proceeded as follows. We have considered the 2D system as divided into columns, say from 1 to L (each column has two ''boundary'' sites, which are never allowed to discharge). When a site with energy $E > E_c$ in column i topples, it distributes an energy αE to two randomly chosen sites, one in column $i-1$ and another one in column $i+1$. Moreover, an energy αE is also received by two randomly chosen sites in column i. In this way the notion of position along the direction perpendicular to the columns acquires a meaning and the effect of the boundaries (located at columns 0 and $L+1$) can propagate into the system. At the same time, the random choice of sites within columns should guarantee that we are dealing with an inhomogeneous mean-field model. The set of all columns behaves as a finite chain with boundaries, from which bulk behavior can also be extracted in the $L \rightarrow \infty$ limit.

Boundary conditions are determined by setting the level of conservation of the dynamics α at sites close to the boundaries (α_{BC}). Two different possibilities are suggested by the Burridge-Knopoff model of earthquakes [14]: (a) *free* boundary conditions, in which $\alpha_{BC} = \alpha/1 - \alpha$ and (b) *open* boundary conditions, in which $\alpha_{BC} = \alpha$. The Burridge-Knopoff model is a driven spring-block model, which can be directly mapped into the OFC model [2]. It can be schematized as a two-dimensional network of blocks interconnected by springs. In addition, all the blocks are subject to an external driving force, which pulls them, and to a static friction, which opposes their motion. The case of free boundary conditions corresponds to boundary blocks connected only to blocks belonging to the system. The case of open boundary conditions, instead, corresponds to blocks at the boundary coupled also to an imaginary external block (see Ref. [2] for further details). Although both of them are probably not realistic, it is believed that more adequate boundary conditions should somehow interpolate between these two extreme limits [15]. Note, moreover, that free boundary conditions are more conservative than open ones and become strictly conservative for $\alpha = \frac{1}{4}$. Below we will show that this distinction leads to a radical difference in the distributions of avalanches propagating from the borders of the chain.

We performed extensive numerical simulations in order to sample bulk and boundary avalanche size distributions. The bulk avalanche distribution of the system appears not to be affected by the choice of boundary conditions. Figures 1(a) and 2(a) report the results in the conservative case ($\alpha = 0.25$) for free and open boundary conditions, respectively. Only avalanches starting from the deep interior of the lattice have been taken into account in the statistics. As a matter of fact, bulk behavior is also easily extracted upon, e.g., averaging the avalanche distribution over all possible starting columns in the lattice. The estimated exponents are $\tau = 1.45 \tau = 1.45 \pm 0.1$ and $\tau = 1.5 \pm 0.1$ for free and open boundary conditions, respectively. In both cases they are consistent with the usual mean-field one, i.e., $\tau = \frac{3}{2}$. In contrast, avalanches starting from the borders, i.e., from columns 1 or L, are strongly influenced by the choice of boundary conditions. Boundary avalanches are distributed as $P(s) \propto s^{-\tau'}$, where $\tau' = 1.4 \pm 0.1$ for free boundaries [Fig. 1(b)] and $\tau' = 1.75 \pm 0.1$ for open boundaries [Fig. 2(b)]. We have also checked that boundary conditions interpolating between free and open behave as the open case, i.e., $\tau' \simeq 1.75$. In this respect $\tau' \simeq 1.75$ reflects a more robust behavior of the model, appropriate to any nonzero level of dissipation realized at the border.
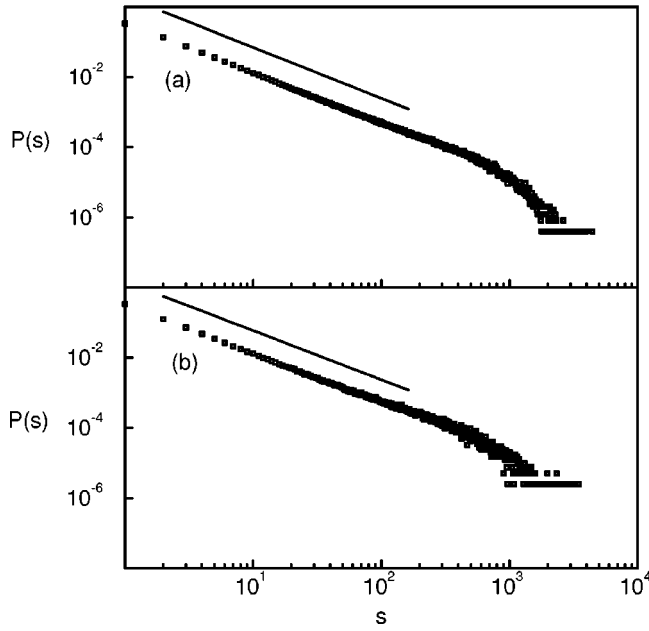
FIG. 1. Simulation results for (a) bulk and (b) boundary avalanches in a system with free boundary conditions. The system size is $L=100$ and the conservation parameter is $\alpha=0.25$. The estimated exponents are (a) $\tau=1.45\pm0.1$ and (b) $\tau'=1.4\pm0.1$.
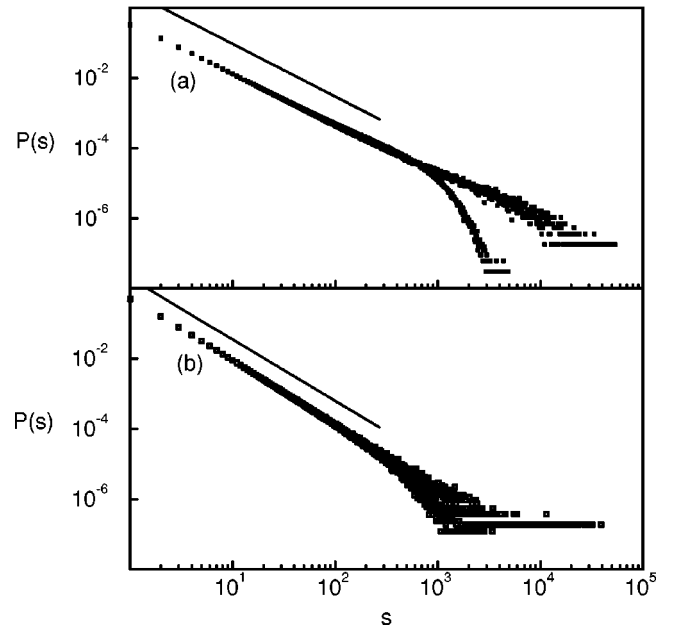

FIG. 2. Results for (a) bulk and (b) boundary avalanches in a system with open boundary conditions. System size are $L=100,200$ and the conservation parameter is $\alpha=0.25$. The estimated exponents are (a) $\tau=1.5\pm0.1$ and (b) $\tau'=1.75\pm0.1$.

Finally we have verified that the introduction of bulk dissipation ($\alpha<1/4$) has no apparent effect on bulk and boundary exponents, as long as $\alpha>\alpha_c$, as determined in Ref. [11]. However, it is controversial [12,13] whether for $\alpha<1/4$ the model should be considered critical. An apparent criticality could result numerically from finite size effects.

Our results indicate that the mean-field OFC model with open boundary conditions is correctly described by an inhomogeneous branching process [16]. This generalization of the standard branching process has been recently proposed and studied as a paradigm for a more complete description of SOC models in the mean-field limit. The inhomogeneous branching process takes place in a situation in which translation invariance is broken. An example is given by the geometry of a semi-infinite chain. Each site of the chain involved in the process at a given stage of the process can activate its neighbors (and/or reactivate itself) at the subsequent stage with well defined probabilities. The probability of generating a tree (or avalanche) with s individuals becomes a function of the position $n=1,\ldots,\infty$ where the tree starts along a chain. It was shown exactly in Ref. [16] that critical trees (avalanches) starting from the boundary are distributed as $P_1(s)\propto s^{-\tau'}$, with $\tau'=\frac{7}{4}$, different from the bulk exponent $\tau=\frac{3}{2}$. Boundary sites were identified there as sites with an average number of branchings smaller than 1 [17], reflecting in this way a sort of dissipative behavior. In this respect open boundary conditions in our model correspond to the existence of these "dissipative" boundaries of the inhomogeneous branching process. On the other hand, if the average number of branchings is maintained equal to 1 for all sites of the chain (including the end), it is possible to verify numerically that, in the inhomogeneous branching process, the exponent τ' coincides with the bulk one (i.e., $\tau'\simeq\frac{3}{2}$) [18]. This case seems to be realized by our model with free

boundary conditions. Thus, free conditions here seem adequate to keep at its critical bulk value (i.e., 1) the average number of descendants for each generations occurring at the border, in the branching process underlying earthquake dynamics.

In conclusion, we have shown how it is possible to introduce proper boundary effects at the level of a RN OFC model. The model we have considered can also be interpreted as a 1D nearest-neighbor model with "∞ components." Indeed, the energies in a given column can be thought of as the many components of an energy vector associated to a single site of a 1D chain. Since we have always considered very large L, our approach effectively corresponds to a ∞-components limit of the 1D model.

We have verified that, with open boundary conditions, boundary avalanches in the OFC model are distributed as $P(s)\propto s^{-7/4}$, as predicted by the analysis of the inhomogeneous branching process. Free boundary conditions, instead, correspond to "conservative" boundary conditions in the inhomogeneous branching process and imply $\tau'\simeq\tau\simeq3/2$.

As a prospective for future work, it would be interesting to investigate whether the mechanism of invasion from the boundaries of the "self-organized" region (named "phase locking" in Ref. [6]) is actually present also in the random neighbor OFC model with boundaries. We expect that also in the nearest-neighbor OFC model, boundary scalings different from the bulk ones should be observed with appropriate conditions at the borders.

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